LETTER TO THE EDITOR

Exact solution of the lattice vertex model analogue of the coupled Bariev XY chains

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Abstract. We present the algebraic Bethe ansatz solution for the vertex model recently proposed by Zhou as the classical analogue of the Bariev interacting XY chains. The relevant commutation rules between the creation fields contain the Hecke symmetry pointed out recently by Hikami and Murakami. The eigenvalues of the corresponding transfer matrix are explicitly given.

Some years ago, Bariev formulated a model of interacting XY chains and solved it exactly, in one dimension, by the coordinate Bethe ansatz approach [1]. The model couples two XY models and its Hamiltonian on a lattice of length L can be written as [1–3]

$$H = \sum_{i=1}^{L} \left\{ (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) \exp(\alpha \tau_{i+1}^+ \tau_{i+1}^-) + (\tau_i^+ \tau_{i+1}^- + \tau_i^- \tau_{i+1}^+) \exp(\alpha \sigma_i^+ \sigma_i^-) \right\} \qquad (1)$$

where $\sigma_i^\pm$ and $\tau_i^\pm$ are two commuting sets of Pauli matrices acting on site i, and α is the coupling constant. Soon after, Bariev [2] generalized this model to include many coupled XY chains. After a Jordan–Wigner transformation, the model (1) and its generalizations can be seen as an electronic system where the hopping term depends (asymmetrically) on the occupation number of the site itself [4, 5]. In this sense, the coordinate Bethe ansatz solution of these models has been used in [4, 5] to explore the finite-size behaviour and the excitations, as well as some related conductivity properties.

  The quantum integrability of (1), however, has only been recently proved by Zhou [3] in terms of the quantum inverse scattering approach. Zhou [3] was able to construct the two-dimensional vertex model whose transfer matrix commutes with the Bariev Hamiltonian (1). The purpose of this letter is to show that such an underlying vertex model can be diagonalized by the algebraic Bethe ansatz [6–8]. We recall that this method is a powerful mathematical technique, which can provide us with information concerning the properties of the Bariev chain within a unified perspective. We also remark that this technique can in principle be useful in the formulation of the problem of computing lattice correlation functions [8, 9]. Our formulation is strongly inspired by our recent construction of the Bethe states of the Hubbard model by means of the quantum inverse scattering approach [10]. We note, however, that the structure of Zhou's R-matrix is a bit different than that appearing in the Hubbard model [11, 12]. Indeed, we shall see that the appropriate parametrization of

the quantum R-matrix associated to the Bariev chain is different and, in fact, much simpler than that which we have found for the Hubbard model [10].

The quantum R-matrix $R(\lambda, \mu)$ found by Zhou [3] has 15 non-zero Boltzmann weights, and following [3] we denote them by $\rho_i(\lambda, \mu)$, $i = 1, \ldots, 15$. However, many of the weights are related to each other under some functional properties, such as $\rho_i(\lambda, \mu) = -\rho_j(\mu, \lambda)$; $\rho_i(\lambda, \mu) = \rho_j(h\lambda, h\mu)$; $\rho_i(\lambda, \mu) = h\rho_j(\lambda, \mu)$†. The parameter h is given in terms of the coupling constant α by $h = \exp(\alpha)$. For explicit expressions we refer readers to [3]. We remark that we have checked that Zhou's R-matrix satisfies explicitly the Yang–Baxter equation. To this purpose, we find it convenient to perform the re-scaling $\lambda \rightarrow \lambda/\sqrt{h}$ and $\mu \rightarrow \mu/\sqrt{h}$, bringing the weights to be slightly more symmetrical as a function of h ($h \rightarrow 1/h$). Here we would also like to quote only a few extra identities which we found relevant in the course of our calculations. These are given by

$$\rho_{15}(\lambda, \mu)[\rho_9(\lambda, \mu) + \rho_1(\lambda, \mu)] = \rho_5(\lambda, \mu)\rho_6(\lambda, \mu), \quad \rho_6(\lambda, \mu)\rho_1(\lambda, \mu) + \rho_5(\lambda, \mu)\rho_{15}(\lambda, \mu)$$
$$= \rho_6(\lambda, \mu)\rho_7(\lambda, \mu) \tag{2}$$

$$\rho_{12}(\lambda, \mu)[\rho_9(\lambda, \mu) + \rho_1(\lambda, \mu)] = \rho_5(\lambda, \mu)\rho_4(\lambda, \mu), \quad \rho_5(\lambda, \mu)\rho_1(\lambda, \mu) + \rho_{15}(\lambda, \mu)\rho_6(\lambda, \mu)$$
$$= \rho_5(\lambda, \mu)\rho_{10}(\lambda, \mu). \tag{3}$$

In order to diagonalize the transfer matrix of the classical vertex model corresponding to the Bariev chain we can basically follow the main steps of our recent algebraic construction of the Bethe ansatz for the Hubbard model [10]. We take as the reference state $|0\rangle$, the standard ferromagnetic vacuum where all the spins are in the 'up' eigenstate of σ_j^z and τ_j^z. We solve the Yang–Baxter algebra for Zhou's R-matrix by writing the monodromy matrix $\mathcal{T}(\lambda)$ in the auxiliary space as

$$\mathcal{T}(\lambda) = \begin{pmatrix} B(\lambda) & \boldsymbol{B}(\lambda) & F(\lambda) \\ \boldsymbol{C}(\lambda) & \hat{A}(\lambda) & \boldsymbol{B}^*(\lambda) \\ C(\lambda) & \boldsymbol{C}^*(\lambda) & D(\lambda) \end{pmatrix} \tag{4}$$

where $\boldsymbol{B}(\lambda)(\boldsymbol{B}^*(\lambda))$ and $\boldsymbol{C}(\lambda)(\boldsymbol{C}^*(\lambda))$ are two component vectors with dimensions $1 \times 2(2 \times 1)$ and $2 \times 1(1 \times 2)$, respectively. The operator $\hat{A}(\lambda)$ is a 2×2 matrix and the other remaining operators are scalars. The transfer matrix $T(\lambda)$ is the trace of $\mathcal{T}(\lambda)$ on the auxiliary space, and the eigenvalue problem becomes

$$\left[B(\lambda) + \sum_{a=1}^{2} A_{aa}(\lambda) + D(\lambda) \right] |\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle = \Lambda(\lambda, \{\lambda_i\})|\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle. \tag{5}$$

The set of variables $\{\lambda_1, \ldots, \lambda_n\}$ parametrizes the multiparticle Hilbert space by the action of the creation fields on the reference state $|0\rangle$. The operators $B(\lambda)$, $\boldsymbol{B}^*(\lambda)$ and $F(\lambda)$ play the role of creation fields while $\boldsymbol{C}(\lambda)$, $\boldsymbol{C}^*(\lambda)$, $C(\lambda)$ and $A_{ab}(\lambda)$, for $a \neq b = 1, 2$, are annihilators. This means that the monodromy matrix (4) has a triangular form when acting on the reference state $|0\rangle$. In addition, we have the following 'diagonal' identities

$$B(\lambda)|0\rangle = |0\rangle \qquad D(\lambda)|0\rangle = [\lambda]^{2L}|0\rangle \qquad A_{aa}(\lambda)|0\rangle = [\lambda h]^{L}|0\rangle \qquad a = 1, 2. \tag{6}$$

A crucial step in algebraically solving the eigenvalue problem (5) is to find the appropriate commutation rules between two fields of $B(\lambda)$ or $\boldsymbol{B}^*(\lambda)$ type. Similar to what happens for the Hubbard model [10], their commutation rules are equivalent, because they generate only the common creation field $F(\lambda)$ as a new operator. Remarkably enough, these commutation rules already encode the basic underlying hidden symmetry of the Bariev

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† For instance, from [3], it is possible to check that $\rho_{11}(\lambda, \mu) = \rho_9(\lambda/h, \mu/h)$; $\rho_4(\lambda, \mu) = \rho_2(\lambda, \mu)\rho_2(\lambda/h, \mu/h)$; $\rho_5(\lambda, \mu) = -\rho_{12}(\mu, \lambda)$.

chain [13]. We can see this, for instance, in the commutation relation† between the fields $B(\lambda)$ and $B(\mu)$

$$B(\lambda) \otimes B(\mu) = B(\mu) \otimes B(\lambda)\hat{r}(\lambda, \mu)$$
$$+ \frac{\xi}{\rho_9(\lambda, \mu)} \{\rho_5(\lambda, \mu)F(\lambda)B(\mu) + \rho_5(\mu, \lambda)F(\mu)B(\lambda)\} \tag{7}$$

where the vector ξ and the matrix $\hat{r}(\lambda, \mu)$ have the following structures

$$\xi = (0 \quad 1 \quad h^{-1} \quad 0) \qquad \hat{r}(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(\lambda, \mu) & b(\lambda, \mu) & 0 \\ 0 & b(\lambda, \mu) & \tilde{a}(\lambda, \mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

and functions $a(\lambda, \mu)$, $\tilde{a}(\lambda, \mu)$ and $b(\lambda, \mu)$ are given by

$$a(\lambda, \mu) = \frac{\lambda(1 - h^2)}{\lambda - h^2\mu} \qquad \tilde{a}(\lambda, \mu) = \frac{\mu(1 - h^2)}{\lambda - h^2\mu} \qquad b(\lambda, \mu) = -\frac{h(\lambda - \mu)}{\lambda - h^2\mu}. \tag{9}$$

The structure of the Boltzmann weights of the matrix $\hat{r}(\lambda, \mu)$ are the same as that appearing in the six-vertex model with an azimuthal anisotropy η given by $\eta = i\ln(h) = i\alpha$. In order to see this, we have to introduce the following parametrization,

$$\lambda = \exp[ik(\lambda)] \tag{10}$$

and consequently the Boltzmann weights can be rewritten in terms of the difference $k = k(\lambda) - k(\mu)$ and the anisotropic constant η as

$$a(k) = \frac{\exp(ik/2)\sin(\eta)}{\sin(k/2 + \eta)} \qquad \tilde{a}(k) = \frac{\exp(-ik/2)\sin(\eta)}{\sin(k/2 + \eta)} \qquad b(k) = -\frac{\sin(k/2)}{\sin(k/2 + \eta)}. \tag{11}$$

By means of a transformation which preserves the Yang–Baxter equation, the so-called symmetry breaking transformation [14], the Boltzmann weights $a(k)$ and $\tilde{a}(k)$ can be symmetrized in order to give the standard anisotropic six-vertex model. In fact, the presence of the asymmetric version of the six-vertex model in the commutation rules is a clear sign that the underlying symmetry is of Hecke type. It is not difficult to see, from the asymmetric vertex model (8), that we can construct a braid operator which appears as the generator of the Hecke algebra [14]. We remark here that such symmetry was first noticed by Hikami and Murakami [13] in the context of the lattice Schrödinger equation for the 'fermionic' formulation of the Bariev Hamiltonian (1). The only subtle point is the minus sign on the weight $b(k)$. The Yang–Baxter equation is invariant under $b(k) \rightarrow -b(k)$, and the sign $\pm$ can be interpreted as the periodic/antiperiodic boundary conditions when the size of the quantum Hilbert space is odd [15]‡.

Now, the construction of the eigenvectors and the eigenvalues goes fairly parallel to the formulation we have recently presented for the Hubbard model [10]. The only subtlety here is that the vertex operator associated with the nested problem (see equation (8)) is asymmetric. We remark that one has to consider carefully this property in order to show, for example, that the unwanted terms coming from the two-particle state are indeed cancelled out. The final answer, however, for the eigenvalues and Bethe ansatz equations depends only on the weight $b(k)$. In order to see this, we basically have to adapt the 'diagonal' commutation rules of [10] by taking into account the specific weights of Zhou's R-matrix and also considering the convenient parametrization given in equation (10). Here we only

† We note that identities such as (2) and (3) are important in the simplification of the commutation rules.
‡ For an even size the sign does not matter. We also recall that in the fermionic formulation of the Bariev chain this sign is positive.

present our final results for the eigenvalues of the 'covering' vertex model. Many other results, as well as the main technical steps we have developed will be presented in a separate publication [15], together with the detailed algebraic solution of the Hubbard model [10]. We remark that, recently, the exact expression for the eigenvalue appears to be very important in the study of finite-temperature properties of integrable models [16–18]. We found that the eigenvalue of the transfer matrix associated with the Bariev chain is given by

$$\Lambda(\lambda, \{\lambda_i\}) = \prod_{i=1}^{n} \frac{h^{-1} + h\lambda_i\lambda}{\lambda_i - \lambda} + \lambda^{2L} \prod_{i=1}^{n} \frac{1 + h^2\lambda_i\lambda}{\lambda - h^2\lambda_i} + [\lambda h]^L \prod_{i=1}^{n} \frac{h^{-1} + h\lambda_i\lambda}{\lambda - \lambda_i} \Lambda^{(1)}(\lambda, \{\lambda_i\})$$

(12)

where $\Lambda^{(1)}(\lambda, \{\lambda_i\})$ is the eigenvalue of the vertex model defined by the auxiliary R-matrix $\hat{r}(\lambda, \mu)$ in the presence of inhomogeneities. Furthermore, the variables $\{\lambda_i\}$ are constrained by the Bethe ansatz equation

$$[\lambda_i h]^{-L} = -(-1)^n \Lambda^{(1)}(\lambda = \lambda_i, \{\lambda_j\}) \qquad i = 1, \ldots, n.$$

(13)

The auxiliary problem can be solved by using the standard six-vertex formulation of Faddeev *et al* [6, 7], adapted to include the inhomogeneities $\{\lambda_i\}$. In the diagonalization procedure, it is necessary to introduce the auxiliary variables $\{\mu_j\}$ and the eigenvalue $\Lambda^{(1)}(\lambda, \{\lambda_i\}, \{\mu_j\})$ reads

$$\Lambda^{(1)}(\lambda, \{\lambda_i\}, \{\mu_j\}) = \prod_{j=1}^{m} \frac{1}{b(\mu_j, \lambda)} + \prod_{i=1}^{n} b(\lambda, \lambda_i) \prod_{j=1}^{m} \frac{1}{b(\lambda, \mu_j)}$$

(14)

where the variables $\{\mu_j\}$ satisfy the equation

$$\prod_{i=1}^{n} b(\mu_j, \lambda_i) = -\prod_{k=1}^{m} \frac{b(\mu_j, \mu_k)}{b(\mu_k, \mu_j)} \qquad j = 1, \ldots, m.$$

(15)

Finally, all these results can be combined in order to give us the eigenvalue and the Bethe ansatz equations. At this point, to cast the final results in a convenient form, we redefine the variables λ, $\{\lambda_i\}$, and $\{\mu_j\}$ by

$$\lambda_i h = \exp(ik_i) \qquad \mu_j = \exp(i\Lambda_j) \qquad \lambda = \exp(ik).$$

(16)

In terms of these new parameters the expression for the eigenvalue is

$$\Lambda(k, \{k_i\}, \{\Lambda_j\}) = \prod_{i=1}^{n} \frac{\cos(k/2 + k_i/2 - \eta/2)}{i\sin(k_i/2 - k/2 + \eta/2)} + \exp(i2Lk) \prod_{i=1}^{n} \frac{\cos(k_i/2 + k/2 - \eta/2)}{i\sin(k/2 - k_i/2 + \eta/2)}$$

$$+ \exp[i(k - \eta)L] \left\{ \prod_{i=1}^{n} \frac{i\cos(k/2 + k_i/2 - \eta/2)}{\sin(k_i/2 - k/2 + \eta/2)} \prod_{j=1}^{m} - \frac{\sin(\Lambda_j/2 - k/2 + \eta)}{\sin(\Lambda_j/2 - k/2)} \right.$$

$$+ \prod_{i=1}^{n} \frac{i\cos(k/2 + k_i/2 - \eta/2)}{\sin(k/2 - k_i/2 + \eta/2)} \prod_{j=1}^{m} - \frac{\sin(k/2 - \Lambda_j/2 + \eta)}{\sin(k/2 - \Lambda_j/2)} \right\}$$

(17)

and the nested Bethe ansatz equations are given by

$$\exp(ik_i L) = -(-1)^{n-m} \prod_{j=1}^{m} \frac{\sin(k_i/2 - \Lambda_j/2 + \eta/2)}{\sin(k_i/2 - \Lambda_j/2 - \eta/2)} \qquad i = 1, \ldots, n$$

(18)

$$(-1)^n \prod_{i=1}^{n} \frac{\sin(\Lambda_j/2 - k_i/2 - \eta/2)}{\sin(\Lambda_j/2 - k_i/2 + \eta/2)} = -\prod_{k=1}^{m} \frac{\sin(\Lambda_j/2 - \Lambda_k/2 - \eta)}{\sin(\Lambda_j/2 - \Lambda_k/2 + \eta)} \qquad j = 1, \ldots, m.$$

(19)

This last equation is similar to that found early by Bariev [1, 2] in the context of the coordinate Bethe ansatz approach, as it should be. In order to recover the results of Bariev [1, 2] for the eigenenergies $E(L)$ of the Hamiltonian (1), we just have to take the logarithmic derivative of the transfer matrix eigenvalue at the point $\lambda = 0$. This calculation leads us to

$$E(L) = 2h \sum_{i=1}^{L} \cos(k_i). \tag{20}$$

We conclude this letter with the following remarks. The extra signs we have found in the Bethe ansatz equations (18) and (19) are typical of the 'bosonic' formulation (1) of the Bariev chain. They can be related to peculiar boundary conditions [10, 19], and they are not present if one formulates the diagonalization problem for the 'fermionic' version of (1). Our algebraic formulation has an invariance under $h \to h^{-1}$, which is in accordance with the symmetry of the Bariev chain ($\alpha \to -\alpha$) [1]. Following the results of Shiroischi and Wadati [20], there exists a way of generating a generalized Bariev chain from Zhou's R-matrix. Defining the vertex operator [20] $\mathcal{L}^{\theta_0}(\lambda) = PR(\lambda, \theta_0)$, where P is a permutator and $R(\lambda, \theta_0)$ is Zhou's R-matrix, we can define a one-parameter (θ_0) family of vertex models by the transfer matrix $T^{\theta_0}(\lambda) = \mathrm{Tr}_a[\mathcal{L}^{\theta_0}_{aL}(\lambda) \cdots \mathcal{L}^{\theta_0}_{a1}(\lambda)]$. Such a vertex model can be diagonalized following the basic steps we have presented so far. The main change is concerned with the action of the 'diagonal' operators on the reference state $|0\rangle$. In this case, we find $B(\lambda)|0\rangle = [\rho_1(\lambda, \theta_0)]^L|0\rangle$, $A_{aa}(\lambda)|0\rangle = [\rho_3(\lambda, \theta_0)]^L|0\rangle$, and $D(\lambda)|0\rangle = [\rho_9(\lambda, \theta_0)]^L|0\rangle$, in such a way that only the terms which are proportional to the power of L change in the expressions (12) and (13). Finally, since the parametrization (10) is quite simple, it seems interesting to re-investigate the Yangian symmetry as well as the analytical properties of the transfer matrix associated with the Bariev chain in light of the recent results of [21, 22]. Very recently, we have noticed that a second R-matrix formulation (due to the freedom present in free-fermion models) of the Bariev chain is also possible [23]. It also seems interesting to see how the algebraic Bethe ansatz approach works for such different embedding.

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